NRL Report 7797

Linear Integer Programming

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Mathematics Research Center
Report 74-1

Mathematics Information Sciences Division

December 31, 1974

NAVAL RESEARCH LABORATORY
Washington, D.C.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Report 7797	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) LINEAR INTEGER PROGRAMMING		5. TYPE OF REPORT & PERIOD COVERED Interim report on a continuing problem
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) James J. Swanek		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, D.C. 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NRL Problem B01-11 RR 014-02-41-6153
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, Va. 22217		12. REPORT DATE December 31, 1974
		13. NUMBER OF PAGES 47
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Abelian group
Abelian polyhedron
Computer implementation
Cone
Convex set
Face
Free group
Group theoretic method
Half-space
Linear integer programing
Polyhedral set
Supporting hyperplane
Vertex

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Some of the mathematical aspects of the group theoretic approach to linear integer programs were analyzed. The manner in which an Abelian group is associated with a linear integer program has been analyzed. Corresponding to each linear integer program is an associated polyhedron. The geometric structure of this polyhedron reflects the structure of the linear integer program. If the polyedron is known, then under appropriate conditions the optimal solution of the integer program is known. Some properties of these Abelian polyhedra, associated with Abelian groups, are discussed. A computer implementation of the group theoretic approach is included in the report.

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S/N 0102-014-6601 |

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CONTENTS

LINEAR INTEGER PROGRAMMING

THE PROBLEM

The following problem provided the motivation for this study:

$$\text{Maximize} \quad z = cx$$

subject to

$$Ax \leqslant b$$

$$x \geqslant 0, \text{ and } x \text{ is an integer}$$

where

A is an $m \times n$ matrix

b is a $m \times 1$ column vector

x is a $n \times 1$ column vector

c is a $1 \times n$ row vector.

This is a standard linear integer programming problem. Problems of this form occur quite frequently in modern technology and in numerous applications. In many cases the ease of formulation is quite deceptive because they are in general very difficult to solve numerically.

During the last decade a powerful new method, the group theoretic approach, was developed. This novel approach was pioneered by several research workers: Glover [1], Gomory [2], Hu [3], and White [4].

This report is concerned with some of the mathematical aspects of the group theoretic approach to linear integer programming. The manner in which a group is associated with a linear integer program is discussed. Corresponding to each linear integer program is an associated polyhedron. The geometric structure of this polyhedron reflects the structure of the linear integer program. More precisely, if one knows the polyhedron, then under appropriate conditions the optimal solution of the integer program is known. For algebraic reasons, these polyhedra will be called Abelian polyhedra. Some properties of Abelian polyhedra associated with Abelian groups are considered. Finally, a computer implementation of the group theoretic approach is presented in Appendix A.

Note: Manuscript submitted July 10, 1974.

THE GROUP OF A LINEAR INTEGER PROGRAM

Consider the integer program:

$$\text{Maximize } z = cx$$

subject to

$$Ax = b$$

$$x \geqslant 0, \text{ and } x \text{ is an integer}$$

where A is an $m \times n$ matrix.

We may assume that A contains the $m \times m$ identity matrix I_m as a submatrix. This is not too restrictive because we can add slack variables to a set of less-than-or-equal-to inequalities, and any linear program can be expressed in terms of less-than-or-equal-to inequalities. In addition, assume that A and b contain only integer entries.

Suppose an optimal, not necessarily integral, solution $\hat{x} = (\hat{x}_B, \hat{x}_N)$ has been obtained with optimal basis B where $\hat{x}_B$ and $\hat{x}_N$ are the basic and nonbasic variables, respectively. The problem is reformulated as

$$\text{Maximize } z = c\hat{x} = c_B\hat{x}_B + c_N\hat{x}_N$$

where

$$A\hat{x} = B\hat{x}_B + N\hat{x}_N = b$$

$$\hat{x} \geqslant 0.$$

Since

$$\hat{x}_B = B^{-1}(b - N\hat{x}_N),$$

then

$$\begin{aligned} c\hat{x} &= c_BB^{-1}(b - N\hat{x}_N) + c_N\hat{x}_N \\ &= c_BB^{-1}b - (c_BB^{-1}N - c_N)\hat{x}_N \\ &= c_BB^{-1}b - c_N^*\hat{x}_N \end{aligned}$$

where $c_N^* = c_BB^{-1}N - c_N \geqslant 0$ is the relative cost of $\hat{x}_N$.

The problem is to find a nonnegative integer vector $x^I = \left(x_B^I, x_N^I\right)$ that maximizes $z = cx$ and satisfies the constraints. Note that maximizing $cx = c_BB^{-1}b - c_N^*x_N$ with $c_N^* \geqslant 0$ is equivalent to minimizing $c_N^*x_N$. We may express the corresponding linear integer program in the following form.

Minimize $c_N^* x_N$, where $c_N^* \geqslant 0$ is the fixed relative cost of x_N, subject to

$$Bx_B + Nx_N = b,$$

$$(x_B, x_N) \geqslant 0, \text{ and } (x_B, x_N) \text{ is integral.}$$

Consider the column vectors of B and N as elements of the free Abelian group

$$F_m = \bigoplus_{i=1}^{m} Z,$$

the direct sum of m copies of the integers Z. Because $A = (B, N)$ contains the $m \times m$ identity matrix I_m as a submatrix, the subgroup generated by the columns of A is the entire free group F_m. Let F_B be the subgroup generated by the columns of B. The factor group $G = F_m/F_B$ is called the group of the linear integer program relative to the basis B.

The group G is a finite Abelian group of order equal to the determinant of B, $\mathrm{Det}(B)$. More precisely,

$$G = \bigoplus_{i=1}^{m} Z(\epsilon_i)$$

where

$$\epsilon_i | \epsilon_{i+1} \quad \text{and} \quad \prod_{i=1}^{m} \epsilon_i = \mathrm{Det}(B).$$

That is, G is a direct sum of the group of integers modulo ϵ_i, ϵ_i is a factor of ϵ_{i+1}, and the product of the integers ϵ_i is equal to the order of G, $\mathrm{Det}(B)$. This follows from the observation that for any $m \times m$ integer matrix B, there are unimodular integer matrices U_1 and U_2 such that

$$U_1 B U_2 = \mathbf{D}(\epsilon_i)$$

where $\mathbf{D}(\epsilon_i)$ is a diagonal integer matrix with ith diagonal entry ϵ_i, and $\epsilon_i | \epsilon_{i+1}$.

In group theoretic terms, there exists a basis $\{e_i\}$ for the free group

$$F_m = \bigoplus_{i=1}^{m} Z$$

such that $\{\epsilon_i e_i\}$ is a basis for the subgroup F_B generated by the columns of B. We obtain the following expressions in group theoretic notation:

$$F_m = \bigoplus_{i=1}^{m} Z = \bigoplus_{i=1}^{m} \langle e_i \rangle$$

$$F_B = \bigoplus_{i=1}^{m} \langle \epsilon_i e_i \rangle$$

where $\langle g \rangle$ denotes the subgroup generated by the group element g.

Using basic facts from group theory, we obtain

$$G = \frac{F_m}{F_B} = \frac{\bigoplus_{i=1}^{m} \langle e_i \rangle}{\bigoplus_{i=1}^{m} \langle \epsilon_i e_i \rangle} \simeq \bigoplus_{i=1}^{m} \frac{\langle e_i \rangle}{\langle \epsilon_i e_i \rangle} \simeq \bigoplus_{i=1}^{m} Z(\epsilon_i)$$

where $\simeq$ denotes " is isomorphic to."

Let φ be the natural homomorphism mapping F_m onto the factor group G. Let $\mathbf{b}_i$ and $\mathbf{n}_{i+m}$ be the column vectors of $\mathbf{B}$ and $\mathbf{N}$, respectively. Consider

$$\mathbf{B}x_N + \mathbf{N}x_N = \sum_{i=1}^{m} x_i \mathbf{b}_i + \sum_{i=1}^{n-m} x_{m+i} \mathbf{n}_{m+i} = b$$

where x_i are nonnegative integers and these relations are considered as an equation in the free group F_m. In the next diagram the symbol $\rightarrowtail$ denotes the one-to-one homomorphism i and the symbol $\twoheadrightarrow$ denotes the onto homomorphism φ;

$$F_B \overset{i}{\rightarrowtail} F_m \overset{\varphi}{\twoheadrightarrow} G .$$

Applying φ to the above equation over F_m, we obtain a corresponding equation over G:

$$\sum_{i=1}^{m} x_i \varphi(b_i) + \sum_{i=1}^{n-m} x_{i+m} \varphi(n_{i+m}) = \varphi(b) .$$

Because b_i is in the subgroup F_B generated by the columns of B, which is the kernel of φ, $\varphi(b_i) = 0$ in G. For any group element g in G, define

$$v(g) = \sum_{\varphi(n_{i+m})=g} x_{i+m}$$

if there is a column n_{i+m} of N such that $\varphi(n_{i+m}) = g$, otherwise set $v(g) = 0$. Thus, we obtain the following group equation over G where G^+ denotes $G - \{0\}$:

$$\sum_{g \in G^+} v(g)\, g = g_b$$

where $g_b = \varphi(b)$, $v(g) \geqslant 0$ and $v(g)$ is an integer.

Define

$$\pi(g) = \operatorname*{Min}_{i} \{c^*_{i+m} \mid \varphi(n_{i+m}) = g\}$$

where c^*_{m+i} is the relative cost associated with column n_{i+m} of N. If there does not exist a column n_{i+m} of N such that $\varphi(n_{i+m}) = g$, define $\pi(g)$ as $+\infty$; that is, a large positive number.

To summarize, every linear integer program of the form

$$\text{Maximize } z = cx = c_B x_B + c_N x_N = c_B B^{-1} b - (c_B B^{-1} N - c_N) x_N$$

subject to

$$Ax = Bx_B + Nx_N = b,$$

$$x \geqslant 0, \text{ and } x \text{ is an integer},$$

where A is an $m \times n$ integer matrix with optimal basis B and b is an integer $m \times 1$ column vector, induces the following optimization problem over the finite Abelian group G:

$$\text{Minimize } \sum_{g \in G^+} \pi(g) v(g)$$

where $\pi(g) \geqslant 0$ subject to

$$\sum_{g \in G^+} v(g) g = g_0 .$$

Here, $v(g)$ are nonnegative integers, g are nonzero group elements in G, and g_0 is a fixed group element of G. Under suitable conditions, an optimal solution of this group problem completely determines an optimal solution of the corresponding linear integer programming problem. See Gomory [2] or Hu [3] for details.

GEOMETRIC ASPECTS OF LINEAR INTEGER PROGRAMMING

Consider the linear programming problem:

$$\text{Maximize } z = cx$$

subject to

$$Ax = b$$

$$x \geqslant 0$$

where A and b have integer entries. Assume that $\{x | Ax = b \text{ and } x \geqslant 0\}$ is a bounded set. There are four convex sets that may be associated with this problem that give insight into the geometric structure of the corresponding linear integer program. They are the following:

$P_0 = \{x | Ax = b \text{ and } x \geqslant 0\}$,

$P_I =$ convex hull $\{x | Ax = b,\ x \geqslant 0, \text{ and } x \text{ is an integer}\}$,

$P^B = \{x | Ax = b \text{ and } x_N \geqslant 0\}$, where x_N are the nonbasic variables associated with a fixed optimal basis B,

$P_I^B =$ convex hull $\{x | Ax = b,\ x_N \geqslant 0, \text{ and } x \text{ is an integer}\}$.

The vectors x are elements of Euclidean n space E^n. The convex hull of the set S in E^n will be denoted by Conv(S). The convex hull of a set S is the smallest convex set containing S. The affine hull of a set S will be denoted by Aff(S). The affine hull of a set is the smallest affine subspace containing the set.

The set P_I^B is called the corner polyhedron associated with the optimal basis B of the linear program. The geometry of this set P_I^B plays an important role in analyzing the behavior of the associated linear integer program. The various relations between these sets are illustrated by Fig. 1.

Note that the optimal solutions of the linear program correspond to vertices of the convex polyhedron P_0. The optimal solutions of the integer program correspond to vertices of the polyhedron P_I. See Gomory [2] for a more detailed discussion of the preceding points.

ABELIAN POLYHEDRA

In the following discussion familiarity with basic facts from elementary group theory will be assumed. Some acquaintance with Gomory's work [2] on the group theoretic approach to integer programming would be helpful. It will be assumed that all Abelian groups referred to are finitely generated and any references to a vector space will be to Euclidean n-space E^n. Some definitions are required.

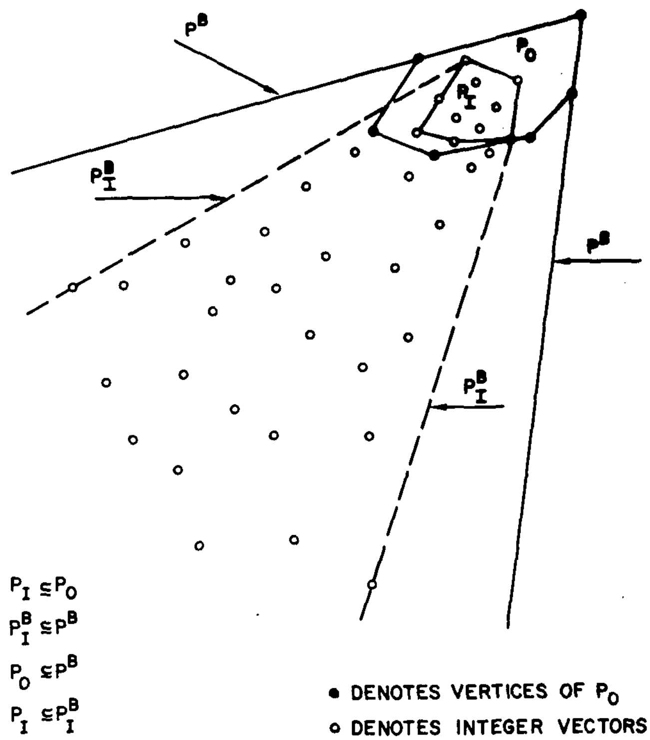

Fig. 1—Diagram showing the relationships between P_0, P_I, P^B, and P_I^B

Definition 1. Hyperplane $\langle a, x\rangle = \beta$

The hyperplane $\langle a, x\rangle = \beta$ is the set $\{x \in E^n | \langle a, x\rangle = \beta\}$ where a is a fixed vector in E^n and x is an arbitrary vector in E^n. The symbol

$$\langle a, x\rangle = \sum_{i=1}^{n} a_i x_i$$

denotes the inner product of the vectors a and x, β is a scalar.

Definition 2. Half-Space $H^+ = \langle a, x\rangle \leq \beta$ or $H^- = \langle a, x\rangle \geq \beta$

The half-space $H^+ = \langle a, x\rangle \leq \beta$ is the set $\{x \in E^n | \langle a, x\rangle \leq \beta\}$ while the half-space $H^- = \langle a, x\rangle \geq \beta$ is the set $\{x \in E^n | \langle a, x\rangle \geq \beta\}$.

Definition 3. Polyhedral Set

A polyhedral set is a nonempty set in E^n that is the intersection of a finite number of half-spaces.

Definition 4. Polyhedron

A polyhedron is a bounded polyhedral set or, equivalently, the convex hull of a finite set in E^n.

Corresponding to a subgroup K of a free Abelian group F_n on n-generators is a class of convex subsets of E^n. We shall call $G(K, n)$ the Gomory class of F_n relative to the subgroup K. The members Γ_k of $G(K, n)$ will be called Gomory sets of F_n relative to K. Consider the representation of the free group F_n on n-generators as the set of all n-vectors v with integer components v_i. Let $S = \{v \in F_n | v_i \geqslant 0 \text{ and } v \neq 0\}$, and let $\{K + v | v \in F_n\}$ be the coset decomposition of F_n relative to the subgroup K.

Definition 5. Gomory set Γ_k

A Gomory set Γ_k is the convex hull of $(K + v) \cap S$ in E^n for some coset $K + v$ where the elements of $(K + v) \cap S$ are considered to be vectors in E^n.

To every linear integer program there is an associated Gomory set whose geometric structure gives information about the optimal solution of the associated integer program. In certain situations the vertices of the associated Gomory set determine the optimal solution of the linear integer program. In addition, Gomory suggests that a further analysis of the geometric structure of Gomory sets could lead to the development of improved methods for solving linear integer programs. In terms of the original integer program we are interested in the Gomory set generated by the coset $F_B + \mathbf{b}$ where F_B is the group generated by the columns of the optimal basis B and $\mathbf{b}$ is the right-hand requirement vector. Thus, it is of some interest to investigate the vertices of a Gomory set.

Definition 6. Vertex of a convex set

A vertex of a convex set is an element of the set that cannot be expressed as a convex combination of other elements of the set.

Determining all vertices of a given Gomory set Γ_k appears to be quite difficult in general. Some insight may be gained by reformulating this problem in algebraic terms. We shall need the following definitions. Note that there is a partial ordering on F_n defined by $v^1 \leqslant v^2$ if and only if for each component i, $v_i^1 \leqslant v_i^2$.

Definition 7. Minimal element of $(K + \hat{v}) \cap S$

A minimal element v° of $(K + \hat{v}) \cap S$ has the properties

(i) $v^\circ \in (K + \hat{v}) \cap S$

(ii) $v \in (K + \hat{v}) \cap S$ and $v \leqslant v^\circ$ implies $v = v^\circ$.

Definition 8. Irreducible element of $(K + \hat{v}) \cap S$

An irreducible element v^* of $(K + \hat{v}) \cap S$ has the properties

(i) $v^* \in (K + \hat{v}) \cap S$

(ii) If $v^1 \leqslant v^*$, $v^2 \leqslant v^*$, and v^1 and v^2 belong to the same set $(K + \widetilde{v}) \cap S$ for some $\widetilde{v}$, then $v^1 = v^2$.

Definition 9. Algebraic extremal of $(K + \hat{v}) \cap S$

An algebraic extremal $\bar{v}$ of $(K + \hat{v}) \cap S$ has the properties

(i) $\bar{v} \in (K + \hat{v}) \cap S$

(ii) If $n\bar{v} = \sum_{j=1}^{n} v^j$ where $v^j \in (K + \hat{v}) \cap S$ and $n > 0$, then $\bar{v} = v^j$ for some j.

The following theorem relates the vertices of a Gomory set $\Gamma_k = \mathrm{Conv}((K + \hat{v}) \cap S)$ to the algebraic extremals of $(K + \hat{v}) \cap S$. First, we require the following lemma.

Lemma 1. *Assume*

$$v^* = \sum_{j=1}^{N} \lambda_j v^j,$$

the components of v^* *and* v^j *are rational numbers, the scalars* λ_j *are positive, and* $\Sigma\lambda_j = 1$. *Then the scalars* λ_j *can be chosen to be rational numbers.*

Proof. Embed the n component vectors v^*, v^j in E^{n+1} by adding an $n + 1$ component equal to 1. Let A be the $n + 1 \times N$ matrix whose jth column is the vector $(v^j, 1)$. Consider the matrix equation

$$A\lambda = (v^*, 1).$$

Assume that this equation has a strictly positive solution $\lambda > 0$. Suppose A has a submatrix B of rank $n + 1$. If $\lambda = (\lambda_B, \lambda_N)$ is partitioned relative to B, then

$$A\lambda = B\lambda_B + N\lambda_N = (v^*, 1)$$

$$B\lambda_B = (v^*, 1) - N\lambda_N .$$

Since the components of λ_N are all positive, there exists a $\hat{\lambda}_N$ with positive rational number components such that $\|\lambda_N - \hat{\lambda}_N\| < \epsilon$ where $\| \ \|$ denotes the Euclidean norm and ϵ is an arbitrary small positive number. Note that B^{-1} has rational number entries since B had rational entries. Continuity considerations imply that the components of $\hat{\lambda}_B$ corresponding to $\hat{\lambda}_N$ are positive rational numbers. Thus, the scalars $\hat{\lambda}_j$ may be chosen as positive rational numbers if the matrix A has rank $n + 1$.

Suppose the rank of $\{(v^j, 1)\}$ is less than $n + 1$, then add the least number of orthonormal basis vectors e^k from E^{n+1} so that the augmented set $\{(v^j, 1)\} \cup \{e^k\}$ has rank $n + 1$. Consider the relations

$$(v^*, 1) + \sum_{k=1}^{\ell} e^k = \sum_{j=1}^{N} \lambda_j (v^j, 1) + \sum_{k=1}^{\ell} e^k$$

$$\frac{1}{\ell + 1}\left[(v^*, 1) + \sum_{k=1}^{\ell} e^k\right] = \frac{1}{\ell + 1}\sum_{j=1}^{N} \lambda_j (v^j, 1) + \frac{1}{\ell + 1}\sum_{k=1}^{\ell} e^k .$$

By the previous case, the scalars on the right may be assumed to be rational numbers r_k:

$$\frac{1}{\ell + 1}\left[(v^*, 1) + \sum_{k=1}^{\ell} e^k\right] = \sum_{j=1}^{N} r_j (v^j, 1) + \sum_{k=1}^{\ell} r_k e^k$$

$$\frac{1}{\ell + 1}(v^*, 1) = \sum_{j=1}^{N} r_j (v^j, 1) + \sum_{k=1}^{\ell} \left(r_k - \frac{1}{\ell + 1}\right) e^k .$$

Let $\langle\{(v^*, 1), (v^j, 1)\}\rangle$ denote the vector subspace of E^{n+1}, generated by the set $\{(v^*, 1), (v^j, 1)\}$. The manner in which the set $\{e^k\}$ was chosen determines that

$$\langle\{(v^*, 1), (v^j, 1)\}\rangle \cap \langle\{e^k\}\rangle = \langle 0\rangle$$

where 0 is the zero vector. Consequently, all the scalars $r_k - 1/(\ell + 1)$ are zero and $\Sigma_{k=1}^{\ell} r_k = \ell/(\ell + 1)$. Thus, the following relations hold:

$$(v^*, 1) = \sum_{j=1}^{N} (\ell + 1) r_j (v^j, 1),$$

$$(\ell + 1) r_j > 0,$$

$$\sum_{j=1}^{N} (\ell + 1) r_j = \frac{\ell + 1}{\ell + 1} = 1 .$$

This completes the proof of the lemma.

Theorem 1. *Let* Γ_k *be the convex hull of* $(\mathrm{K} + \hat{\mathrm{v}}) \cap \mathrm{S}$. *Then* v^* *is a vertex of* Γ_k *if and only if it is an algebraic extremal of* $(\mathrm{K} + \hat{\mathrm{v}}) \cap \mathrm{S}$.

Proof. Suppose v^* is not an algebraic extremal of $(K + \hat{v}) \cap S$. There exists a positive integer p and vectors v^j in $(K + \hat{v}) \cap S$ that satisfy the following equation in the free group F_n:

$$pv^* = \sum_{j=1}^{p} v^j .$$

This equation implies the following vector equation over E^n:

$$\mathbf{v}^* = \sum_{j=1}^{p} \frac{1}{p} v^j .$$

Since v^p is a convex combination of other vectors in Γ_k, it is not a vertex of Γ_k.

Conversely, suppose $\mathbf{v}^*$ is an integer vector in $(K + \hat{v}) \cap S$ that is not a vertex of Γ_k. There exist vectors $\mathbf{v}^j$ in $(K + \hat{v}) \cap S$ and positive scalars λ_j for which

$$v^* = \sum_{j=1}^{N} \lambda_j v^j$$

and

$$\sum \lambda_j = 1 .$$

Since the components of v^* and $\mathbf{v}^j$ are integers, Lemma 1 implies that the positive scalars λ_j can be chosen as rational numbers. Let $\lambda_j = \ell_j/D$ where D is a least common denominator for the set of rational numbers. Multiplying the preceding vector equation by the integer D, we obtain an equation having meaning in the free group F_n:

$$Dv^* = \sum_{j=1}^{N} \ell_j v^j \qquad \sum_{j=1}^{N} \ell_j = D$$

$$Dv^* = \sum_{j=1}^{D} v^j$$

where both v^* and $v^j \in (K + \hat{v}) \cap S$. Consequently, v^* is not an algebraic extremal of $(K + \hat{v}) \cap S$.

The properties of being a minimal element, an irreducible element, or an extremal element of a Gomory set Γ_k are successively stronger conditions. This is shown by the following proposition.

Proposition 1. *Every algebraic extremal is irreducible. Every irreducible element is a minimal element.*

Proposition 1 follows easily from the definitions. However, the converse does not hold. There are minimal elements that are not irreducible, and there are irreducible elements that are not algebraic extremals. This will be more apparent in the sequel. First, we shall proceed in our development.

The nature of the Gomory set $\Gamma_k = \text{Conv}\,((K + \hat{v}) \cap S)$ depends not only on the free group F_n, but more critically on how the subgroup K is embedded in F_n. We shall illustrate this in a more restrictive setting. First, consider the following definitions.

Definition 10. Torsion kernel of a free group

A torsion kernel K is a subgroup of a free group F_n on n-generators such that the factor group F_n/K is a torsion group.

Definition 11. L. P. Kernel of a free group

An L. P. kernel K is a subgroup of a free group F_n on n-generators such that the factor group F_n/K is of cardinality $n + 1$.

The next theorem and succeeding remarks will relate Gomory sets to the group optimization problem and linear integer programming.

Theorem 2. *A Gomory set* $\Gamma_k = Conv\ ((K + \hat{v}) \cap S)$ *associated with a torsion kernel* K *is an* n*-dimensional polyhedral set.*

Proof. It is assumed that K is a subgroup of the free group F_n with n-generators $\{e^j\}$. By hypothesis, there is a positive integer O_j such that $O_j e^j$ is in K. Suppose that $v^* \in (K + \hat{v}) \cap S$; then

$$v^* + O_j e^j \in (K + \hat{v}) \cap S .$$

The set $\{v^*\} \cup \{v^* + O_j e^j\}$ is contained in $(K + \hat{v}) \cap S$ and is affinely independent. Thus, Γ_k contains an n-simplex and consequently Γ_k is an n-dimensional convex set.

The following remarks on convex sets are well known or easy to show. The convex hull of a set of integer vectors is a closed convex set. Any closed convex set is an intersection of closed half-spaces. Consequently, Γ_k is an intersection of half-spaces. Applying arguments similar to those used by Gomory [2, Theorems 6 and 7], one can easily show that Γ_k is an intersection of a finite number of half-spaces. Hence, Γ_k is a polyhedral set. *Remark.* Suppose that K is an L. P. kernel of the free group F_n on n-generators. It is clear that K is necessarily a torsion kernel and the associated Gomory sets Γ_k = Conv $((K + \hat{v}) \cap S)$ are n-dimensional polyhedral sets in E^n. In the sequel we shall consider only L. P. kernels. These kernels are related to the group optimization problem in the following manner. Let G be a finite Abelian group of cardinality $n + 1$. Consider the following diagram

$$K \rightarrowtail^{i} F_n \overset{\varphi}{\twoheadrightarrow} G$$

where φ is the homomorphism induced by associating the generator e_g of F_n with the nonzero group element g of G. Note that the kernel K of φ is an L. P. kernel of F_n and i is an injection map. Consider the Gomory set Γ_k = Conv $((K + \hat{v}) \cap S)$ where $\varphi(\hat{v}) = \hat{g}$. Note that $(K + \hat{v}) \cap S$ consists of all vectors v whose gth components $v(g)$ are nonnegative integers and v satisfies the group equation $\Sigma_{g \in G^+} v(g)g = \hat{g}$ over G. In the sequel we shall consider the Gomory set $\Gamma_{\hat{g}}$ as the convex hull of all vectors **v** whose gth components $v(g)$ are nonnegative integers and **v** satisfies the group equation $\Sigma_{g \in G^+} v(g)g = \hat{g}$.

Let us consider the following group optimization problem:

Minimize

$$\sum_{g \in G^+} v(g)\pi(g)$$

where $\pi(g) \geqslant 0$ is the fixed cost associated with the group element g, subject to

$$\sum_{g \in G^+} v(g)g = \hat{g}$$

$$v(g) \geqslant 0 .$$

Some vertex v^* of the Gomory set $\Gamma_{\hat{g}}$ is a solution of this problem. Conversely, to every vertex v^* of $\Gamma_{\hat{g}}$, there is a cost function π^* such that v^* solves this optimization problem for the cost function π^*. Furthermore, Gomory has shown that the vertices of $\Gamma_{\hat{g}}$ are closely connected with the optimal solutions of linear integer programs.

The next results deal with the rate of growth of the number of vertices of a Gomory set as a function of the group structure of G. The problem of rates of growth of the number of vertices was suggested by Gomory as an area for further work and investigation. This is important because the nature of the Gomory set reflects the structure of the associated linear integer program. For simplicity we shall deal with the Gomory set Γ_0, the convex hull of all nonnegative integer vectors v in E^n such that

$$\sum_{g \in G^+} v(g)g = 0 \qquad \text{and } v \text{ is not the zero vector.}$$

Let $V(G)$ denote the number of vertices of the Gomory set Γ_0 associated with the finite Abelian group G.

Theorem 3. *Suppose* K *is a subgroup of the group* G. *Consider*

$$K \rightarrowtail^{i} G \overset{\varphi}{\twoheadrightarrow} \frac{G}{K}$$

where i *is the injection map and* φ *is the canonical homomorphism onto the factor group* G/K. *Then* $V(G) \geqslant V(G/K)\,|K|$ *where* $|K|$ *denotes the cardinality of the subgroup* K.

Proof. Let g denote a generic element of G, and h a generic element of G/K. Let v be an arbitrary vertex of Γ_0 over G/K. Note that $v(h)$ is a nonnegative integer and $\Sigma_{h \in (G/K)^+} v(h)h = 0$. Define

$$\varphi^{-1}(h) = \{g \in G | \varphi(g) = h\} .$$

In algebraic terminology, $\varphi^{-1}(h)$ is the coset $g_h + K$ in G where $\varphi(g_h) = h$ and g_h is some fixed member of the coset. Each coset $g_h + K$ contains $|K|$ elements.

It is possible to construct a set of $|K|$ vertices over G corresponding to v over G/K. Intuitively, the canonical homomorphism φ is used to pull v back into G. More precisely, consider the following construction. Suppose $v(h) > 0$ and define a vector $\bar{v}$ over the corresponding coset $\varphi^{-1}(h) = g_h + K$ as follows. Choose an arbitrary g_h in $\varphi^{-1}(h) = g_h + K$ and define $\bar{v}(g_h) = v(h)$ and $\bar{v}(g) = 0$ for all other g in $g_h + K$. Repeat this construction for each h in G/K where $v(h) > 0$ and notice that a set of at least $|K|$ vectors $\bar{v}$ corresponding to v are finally obtained. Applying some elementary group theory, it follows that $\Sigma_{g \in G-K}\, \bar{v}(g)g = k$ where k is in the subgroup K. Using some elementary convexity arguments and some facts about homomorphisms, we see $\bar{v}$ to be a vertex of the Gomory set Γ_k over G. By choosing the vectors $\bar{v}$ carefully, it is possible to force k to be the zero element. Thus, to each vertex v of Γ_0 over G/K, there corresponds $|K|$ vertices $\{\bar{v}\}$ of Γ_0 over G. Intuitively the set $\{\bar{v}\}$ are those vertices over G that are mapped by the homomorphism φ onto v over G/K.

$$\begin{aligned}
\sum_{g \in G-K} \bar{v}(g)g = 0 \xrightarrow{\varphi} & \sum_{g \in G-K} \bar{v}(g)\varphi(g) \\
&= \sum_{h \in (G/K)^+} \sum_{g \in \varphi^{-1}(h)} \bar{v}(g)\varphi(g) \\
&= \sum_{h \in (G/K)^+} \sum_{g \in \varphi^{-1}(h)} \bar{v}(g)h \\
&= \sum_{h \in (G/K)^+} \left[\sum_{g \in \varphi^{-1}(h)} \bar{v}(g) \right] h \\
&= \sum_{h \in (G/K)^+} v(h)h = 0 .
\end{aligned}$$

Consequently, there are at least $V(G/K)|K|$ vertices of the Gomory set Γ_0 over G and Theorem 3 follows.

The following two corollaries are easy consequences of Theorem 3.

Corollary 3.1. *Let* Z(p) *be a minimal subgroup of* G *where* p *is a prime. Then*

$$\mathrm{V(G)} \geqslant \mathrm{V}\left(\frac{\mathrm{G}}{\mathrm{Z(p)}}\right) \mathrm{p}.$$

Corollary 3.2. *Let* H *be a maximal subgroup of* G *where* G/H $\simeq$ Z(p) *and* p *is a prime. Then*

$$V(G) \geqslant V(Z(p)|H|.$$

For example, consider the two groups

$$G_1 = Z(2) \oplus Z(5)$$

$$G_2 = Z(2) \oplus Z(4).$$

The following facts on the number of vertices of the Gomory sets Γ_0 for the respective groups are known:

$$V(Z(2)) = 1,$$

$$V(Z(4)) = 4,$$

$$V(Z(5)) = 10,$$

$$V(G_1) = 40,$$

$$V(G_2) = 9.$$

$$V(G_1) \geqslant V(Z(5)) \cdot 2,$$

$$V(G_1) \geqslant V(Z(2)) \cdot 5,$$

$$V(G_2) \geqslant V(Z(4)) \cdot 2,$$

$$V(G_2) \geqslant V(Z(2)) \cdot 4.$$

The problem of determining strong inequalities to gage the rate of growth of the number of vertices as a function of group structure appears to be quite difficult. The following inequality is conjectured for groups G of large order. If G is a group of order

$$\prod_{i=1}^{N} p_i^{e_i}$$

where p_i is a prime, then

$$V(G) \geqslant \prod_{i=1}^{N} V(Z(p_i))^{e_i}.$$

Corollary 3.3. *If* G *is not a direct sum of cyclic groups of order 2, then* $V(G) \geqslant |G|$.

Proof. Suppose $G = Z(p)$ where p is a prime other than 2. The following facts are easy to verify. For each element g° in $Z(p)$, the vector $(\delta_{g,g^\circ} O(g^\circ))$ is a vertex of the Gomory set Γ_0 where δ_{g,g° is the Kronecker delta and $O(g)$ denotes the order of the group element g. In addition, for each g° in $Z(p)$, the vector v satisfying

$$v(g^\circ) = v(-g^\circ) = 1$$

$$v(g) = 0, \quad g \neq g^\circ \text{ or } -g^\circ$$

is a vertex. Consequently,

$$V(Z(p)) \geqslant |Z(p)|.$$

Suppose the corollary is true for all groups of cardinality less than n and $|G| = n$. The group G has a subgroup $Z(p)$, and $G/Z(p)$ is not a direct sum of cyclic groups of order 2. Apply Corollary 3.1 and the inductive hypothesis to obtain

$$V(G) \geqslant V\left(\frac{G}{Z(p)}\right) p \geqslant \left|\frac{G}{Z(p)}\right| p = |G|.$$

Note that for groups G that are direct sums of cyclic groups of order 2, Gomory [2] has shown that the total number of vertices summed over all Gomory sets Γ_g for $g \neq 0$ is asymptotic to

$$|G|^{\log_2 |G| [1+\phi(|G|)]}$$

where $\phi(|G|) \to 0$ as $|G| \to \infty$.

There is an important class of polyhedra associated with the Gomory sets Γ_g. We shall call these polyhedra P_g, Abelian polyhedra. It will be seen that the Abelian polyhedron P_g is contained in Γ_g, and all vertices of Γ_g are vertices of P_g, but not conversely. The following definitions make this concept more precise.

Definition 12. Abelian polyhedron $P_{\hat{g}}$

Let $\hat{g}$ be a fixed element of the finite Abelian group G. An Abelian polyhedron $P_{\hat{g}}$ is the convex hull of all nonnegative integer vectors $\mathbf{v}$ such that

$$\sum_{g \in B^+} \mathbf{v}(g) g = \hat{g} \quad \text{and} \quad \mathbf{v}(g) \leqslant O(g), \quad \text{(and } v \text{ is not the zero vector)}$$

where $O(g)$ denotes the order of the group element g.

Definition 13. Kernel polyhedron P_0

A kernel polyhedron P_0 is an Abelian polyhedron with fixed element $\hat{g}$ equal to the zero group element.

It is possible to describe the relation between the Gomory set $\Gamma_{\hat{g}}$ and its associated Abelian polyhedron $P_{\hat{g}}$ in greater detail. Recall that $\Gamma_{\hat{g}}$ is an n-dimensional polyhedral set. In fact, Gomory has shown that

$$\Gamma_{\hat{g}} = \bigcap_{i=1}^{M} \pi_i^-$$

where

$$\pi_i^- = \{v \in E^n | \langle \pi_i^-, v \rangle \geqslant 1\}$$

or

$$\pi_i^- = \{v \in E^n | \langle \pi_i^-, v \rangle \geqslant 0\}$$

and the components $\pi_i^-(g)$ of π_i^- have other special properties that reflect the structure of G.

The following remarks will indicate that

$$P_{\hat{g}} = \bigcap_{i=1}^{M} \pi_i^- \cap \bigcap_{j=1}^{N} \pi_j$$

where

$$\pi_j = \{v \in E^n | \langle \pi, v \rangle \leqslant 1\}$$

or

$$\pi_j = \{v \in E^n | \langle \pi, v \} \geqslant 1\}.$$

First, consider the following definition and well-known results on polyhedral sets.

Definition 14. Convex cone

A convex cone C is a subset of E^n with the following property. If v_1 and v_2 are vectors in C, then $\lambda_1 v_1 + \lambda_2 v_2$ is in C where λ_1 and λ_2 are nonnegative scalars.

Theorem 4. **[5,6]** *Let* $S = \{v \in E^n | Av \leqslant b\}$ *be a polyhedral set in Euclidean* n-*space. Then* $S = P + C$ *where* P *is a polyhedron and* C *is a convex cone. In particular,* $C = \{v \in E^n | Av \leqslant 0\}$.

Theorem 5. *Let* P *be an* n-*dimensional polyhedral set and* H^+ *a half-space in Euclidean* n-*space. Then every vertex of the polyhedral set* $P \cap H^+$ *is either a vertex of* P *in* H^+ *or the unique point of intersection of the bounding hyperplane* H *of* H^+ *with an edge of* P *that does not lie on* H *[7].*

Lemma 3.1.

$$P_{\hat{g}} = \Gamma_{\hat{g}} \cap \bigcap_{j=1}^{N} \pi_j = \bigcap_{i=1}^{M} \pi_i^{-} \cap \bigcap_{j=1}^{N} \pi_j$$

Proof. Recall that a polyhedral set in E^n is a finite intersection of half-spaces. In addition, a bounded polyhedral set is a convex hull of a finite set of points in E^n and conversely. See McMullen and Shephard [8] for details. Consequently, there are finitely many half-spaces of π_j of the form

$$\pi_j = \{v \in E^n | \langle \pi_j, v\rangle \leqslant 1\}$$

or

$$\pi_j = \{v \in E^n | \langle \pi_j, v\rangle \geqslant 1\}$$

such that

$$P_{\hat{g}} = \Gamma_{\hat{g}} \cap \bigcap_{j=1}^{N} \pi_j .$$

Note that by Theorem 4, the polyhedral set $\Gamma_{\hat{g}}$ is the vector sum of a convex polyhedron $P_{\hat{g}}^*$ and a convex cone $C_{\hat{g}}$. The nature of $P_{\hat{g}}^*$ is difficult to analyze and $P_{\hat{g}}^*$ may be distinct from $P_{\hat{g}}$. Recall that $P_{\hat{g}}$ is the convex hull of all vectors v with nonnegative integer components such that $\Sigma_{g \in G^+} v(g)g = \hat{g}$. It is conjectured that the half-spaces π_j in Lemma 3.1 are of the form

$$\pi_j = \{v \in E^n | \langle \pi_j, v\rangle \leqslant 1\} ;$$

also, $\pi_j(g) = 0$ for $g \neq g_j$ and $\pi(g_j) = 1/O(g_j)$ where $O(g_j)$ is the order of the group element g.

The following results deal with kernel polyhedra. It is reasonable to say that the structure of kernel polyhedra mirrors the structure of the corresponding group. This will be supported in the following development. It is conjectured that the kernel polyhedron will determine the structure of all Abelian polyhedra associated with a given group. Recall that any linear integer program induces an Abelian polyhedron which reflects the structure of the integer program. If one knows the Abelian polyhedron, then, in many cases, the optimal solution of the linear integer program is known. At any rate, kernel polyhedra have some very interesting symmetry properties that shed some light on their geometric structure. First, some definitions are required.

Definition 15. Collapsed kernel polyhedron

Let V be the set of all vectors v in E^n satisfying the following properties:

(i) The gth component $v(g)$ satisfies $O \leqslant v(g) \leqslant O(g)$,

(ii) $\Sigma_{g \in G^+} v(g)g = 0$ in the group G of cardinal $n + 1$,

(iii) v is not the zero vector,

(iv) Some component $v(g)$ is less than $O(g)$.

A collapsed kernel polyhedron P over a group G is the convex hull of the corresponding set V. Recall that $O(g)$ denotes the order of the group element g.

Intuitively, a collapsed kernel polyhedron is the nontrivial or essential part of the corresponding kernel polyhedron. The collapsed polyhedron is easier to work with and more clearly reflects the algebraic structure of its underlying group.

Definition 16. Supporting hyperplane of a convex set

The hyperplane (π, β) is called a supporting hyperplane of the convex set S if

(i) S is contained in one of the half-spaces (π^+, β) or (π^-, β),

(ii) $S \cap (\pi, \beta) \neq \phi$.

Definition 17. *j*-dimensional face of a convex set

A j-dimensional face F of a convex set S is a set of the form $F = S \cap (\pi, \beta)$ where (π, β) is a supporting hyperplane and Aff(F) is of dimension j.

Definition 18. Facet of a convex set

Let S be an n-dimensional convex set. A facet F of the convex set S is an $n - 1$ dimensional face of S.

For example, consider a tetrahedron in 3-space. It is a 3-dimensional convex polyhedron. A vertex is a 0-face, an edge is a 1-face, a side is a 2-face, or facet in this case, and the tetrahedron itself is the unique 3-face.

The following results illustrate some of the interesting symmetries associated with a collapsed kernel polyhedron. First, the following well-known result on polyhedra is required.

Theorem 6. *Let* P *be a convex polyhedron and let* $\mathrm{W} \subseteq \mathrm{V}$ = *vertices of* P. *Then Conv*(W) *is a face of* P *if and only if Aff*(W) $\cap$ *Conv*$(\mathrm{V} - \mathrm{W}) = \phi$ *[8]*.

Definition 19. Conjugate vector over a group

Let v be an n-dimensional vector in E^n indexed by the elements of a group. The conjugate vector $\bar{v}$ of v is a vector whose gth component is $\bar{v}(g) = v(-g)$.

Lemma 3.2. *The faces of a collapsed kernel polyhedron* P *over a group* G *occur in conjugate pairs. If* F *is a* j*-dimensional face of* P, *there exists a* j*-dimensional conjugate face* $\overline{\mathrm{F}}$.

Proof. Let V be the set of vertices of P and the subset $\left\{v_i^F\right\}$ be the vertices of F. Apply Theorem 6. Thus,

$$\mathrm{Aff}\left(\left\{v_i^F\right\}\right) \cap \mathrm{Conv}\left(V - \left\{v_i^F\right\}\right) = \phi .$$

Note that $v = \Sigma\ \lambda_i v^i$ implies $\bar{v} = \Sigma\ \lambda_i \bar{v}^i$, taking conjugates preserves linear combinations. This implies

$$\mathrm{Aff}\left(\left\{\bar{v}_i^F\right\}\right) \cap \mathrm{Conv}\ V - \left(\left\{\bar{v}_i^F\right\}\right) = \phi .$$

Consequently, $\overline{F} = \mathrm{Conv}\left(\left\{\bar{v}_i^F\right\}\right)$ is a j-dimensional face of P.

Corollary 3.4. *Vertices of a collapsed kernel polyhedron occur in conjugate pairs.*

Note that some faces may be self conjugate, that is $F = \overline{F}$. However, conjugation is an involution, $\overline{\overline{F}} = F$.

There is another class of symmetries associated with a collapsed kernel polyhedron. Consider the next definition.

Definition 20. Complement vector over a group

Let v be an n-dimensional vector in E^n indexed by the elements of a group. The complement vector v^c of v is a vector whose gth component is $v^c(g) = O(g) - v(g)$ where $O(g)$ denotes the order of the group element g.

It is easy to see that complementation is an involutory, affine transformation of E^n:

$$v^{cc} = v$$

$v = \Sigma\ \lambda_i v_i$ and $\Sigma\ \lambda_i = 1$ implies $v^c = \Sigma\ \lambda_i v_i^c$.

Lemma 3.3. *The faces of a collapsed kernel polyhedron occur in complementary pairs. If* F *is a* j*-dimensional face, there exists a* j*-dimensional complementary face* $\mathrm{F^c}$.

Proof. The proof is similar to that of Lemma 3.2.

Corollary 3.5. *Vertices of a collapsed kernel polyhedron occur in complementary pairs.*

In contradistinction to the operation of conjugation, there are no self-complementary proper faces. This will be shown in the sequel. The next corollary formalizes this observation.

Corollary 3.6. *A collapsed kernel polyhedron has an even number of proper faces of each dimension.*

If the facets of an Abelian polyhedron are known, then the Abelian polyhedron is known and consequently, certain structural properties of a linear integer program associated with the Abelian polyhedron are known. To further understand how the algebraic structure of the group G is reflected in the geometric structure of the corresponding collapsed kernel polyhedron P, it is necessary to analyze the facets F of P. Recall that the vertices of a facet F span an $n - 1$ dimensional affine subspace and are a subset of the vertices of P. Consequently, a facet F determines its supporting hyperplane (π, β) relative to a multiplicative constant. Thus, the facets of P fall into one of three classes depending on the type of supporting hyperplane. The supporting hyperplanes may be one of the following types:

$$\text{(i)}\ \pi^\circ = \{v \in E^n \,|\, \langle \pi^\circ, v\rangle \geqslant 0\},$$

$$\text{(ii)}\ \pi^+ = \{v \in E^n \,|\, \langle \pi^+, v\rangle \leqslant 1\},$$

$$\text{(iii)}\ \pi^- = \{v \in E^n \,|\, \langle \pi^-, v\rangle \geqslant 1\}.$$

In particular, any facet F is of one of the following forms:

$$F = P \cap \{v \in E^n \,|\, \langle \pi^\circ, v\rangle = 0\},$$

$$F = P \cap \{v \in E^n \,|\, \langle \pi^-, v\rangle = 1\},$$

$$F = P \cap \{v \in E^n \,|\, \langle \pi^+, v\rangle = 1\}.$$

As a matter of notation, all vectors v and π are in E^n relative to an orthonormal basis. These vectors are indexed by group elements g of G, and $v(g)$ or $\pi(g)$ denotes the gth component of the respective vectors.

Lemma 3.4. *Let* $\langle \pi^\circ, \mathrm{v}\rangle \geqslant 0$ *be a supporting hyperplane of a collapsed kernel polyhedron* P *that induces a facet* F. *Then there is a fixed component* g' *such that* $\pi^\circ(\mathrm{g}') = 1$ *and* $\pi^\circ(\mathrm{g}) = 0$ *for* $\mathrm{g} \neq \mathrm{g}'$.

Proof. Recall that P is the convex hull of all nonnegative integer vectors $\mathbf{v}$ such that

$$\sum_{g \in G^+} v(g)g = 0, \qquad \mathbf{v} \neq 0 \qquad \text{and } v(g) < O(g) \text{ for some component } g.$$

Since $\mathrm{Aff}(F)$ is $n - 1$ dimensional, there are $n - 1$ linearly independent vertices of F. Thus, the vertices of F would not all have zero components for two different coordinates g_1 and g_2. This means there is a fixed coordinate g', and for $g \neq g'$ there is a vertex v^* in F and $v^*(g) > 0$. However,

$$\langle \pi^\circ, v^*\rangle = \sum_{g \in G^+} \pi^\circ(g)v^*(g) = 0.$$

Note that $\pi^\circ(g) \geqslant 0$ for all g, since $\langle \pi^\circ, v\rangle \geqslant 0$ supports P and $(\delta_{g',g} O(g))$ is in P where $\delta_{g',g}$ is the Kronecker delta. Combining the two previous observations shows that $\pi^\circ(g) = 0$ for $g \neq g'$ and $\pi^\circ(g')$ can be chosen to be 1.

Lemma 3.5. *Let* $\langle \pi^-, v\rangle \geqslant 1$ *be a supporting hyperplane of a collapsed kernel polyhedron* P *that induces a facet* F. *Then* π^- *satisfies the conditions*

$$\text{(i)}\quad \pi^-(g) \geqslant \frac{1}{O(g)},$$

$$\text{(ii)}\quad \pi^-(g) + \pi^-(-g) = 1,$$

$$\text{(iii)}\quad \pi^-(g_1 + g_2) \leqslant \pi^-(g_1) + \pi^-(g_2) \leqslant 1 + \pi^-(g_1 + g_2).$$

Proof. Note that the vector $(\delta_{g',g} O(g))$ is in P where $\delta_{g',g}$ is the Kronecker delta. Since $\langle \pi^-, v\rangle \geqslant 1$ supports P, $O(g')\pi^-(g') \geqslant 1$ for all $g' \neq 0$ in the group G. Thus, condition (i) holds. Because of the nature of the supporting hyperplane, the facet F contains n linearly independent vertices. This implies that for any coordinate g', there is a vertex v' of F such that $v'(g') > 0$. Note that $v'(g') = O(g')$ would imply that $v'(g) = 0$ for all $g \neq g'$ and $\pi^-(g') = 1/O(g)$. Consider

$$\langle \pi^-, v'\rangle = \sum_{g\in G^+} \pi^-(g)v'(g) = \pi^-(g')v'(g') + \sum_{g\neq g'} \pi^-(g)v'(g)$$

$$= \pi^-(g') + \pi^-(g')[v'(g') - 1] + \sum_{g\neq g'} \pi^-(g)v'(g) = 1.$$

Suppose that $s' = g_1 + g_2$. Define a vector v^* as

$$v^*(g) = v'(g) \qquad g \neq g', g_1, \text{ or } g_2,$$

$$v^*(g') = v'(g') - 1,$$

$$v^*(g_1) = v'(g_1) + 1,$$

$$v^*(g_2) = v'(g_2) + 1.$$

Consequently, v^* is a proper vector, that is $O \leqslant v^*(g) \leqslant O(g)$ for all $g \neq 0$. The vector v^* satisfies $\Sigma_{g\in G^+} v^*(g)g = 0$ and consequently is in P. Since $\langle \pi^-, v\rangle \geqslant 1$ supports P, we have that $\langle \pi^-, v^*\rangle \geqslant 1$. Hence, the first part of condition (iii) holds;

$$\pi^-(s') \leqslant \pi^-(g_1) + \pi^-(g_2).$$

Condition (ii) will be proved next and then the second part of condition (iii) will follow as a consequence.

Assume that $g^\circ \neq -g^\circ$ and define a vector v' as

$$v'(g^\circ) = v'(-g^\circ) = 1$$

$$v'(g) = 0 \qquad\qquad g \neq g^\circ \text{ or } -g^\circ .$$

Note that v' is in P and thus $\langle \pi^-, v' \rangle = \pi^-(g^\circ) + \pi^-(-g^\circ) \geqslant 1$.

Consider a vertex v^* of F such that $v^*(g^\circ) > 0$.

$$\pi^-(g^\circ) + \pi^-(g^\circ)[v^*(g^\circ) - 1] + \sum_{g \neq g^\circ} \pi^-(g)v^*(g) = 1 .$$

The validity of the following group equation is clear.

$$-g^\circ = (v^*(g^\circ) - 1)g^\circ + \sum_{g \neq g^\circ} v^*(g)g .$$

Recall that π^- is subadditive on G. Thus,

$$\pi^-(-g^\circ) \leqslant \pi^-(g^\circ)(v^*(g^\circ) - 1) + \sum_{g \neq g^\circ} \pi^-(g)v^*(g),$$

and consequently $\pi^-(g^\circ) + \pi^-(-g^\circ) \leqslant 1$. Finally, $\pi^-(g^\circ) + \pi^-(-g^\circ) = 1$ for all $g^\circ \neq -g^\circ$ in G.

It can be shown that condition (ii) also holds for elements g° of order 2.

The second part of condition (iii) follows from the easily verified set of relations

$$\pi^-(g_1 + g_2) + \pi^-(-g_1) + \pi^-(-g_2) \geqslant 1,$$

$$\pi^-(g_1) + \pi^-(g_2) + \pi^-(-g_1) + \pi^-(-g_2) = 2,$$

$$[\pi^-(g_1) + \pi^-(g_2) - \pi^-(g_1 + g_2)] + [\pi^-(g_1 + g_2) + \pi^-(-g_1) + \pi^-(-g_2)] = 2,$$

$$\pi^-(g_1) + \pi^-(g_2) - \pi^-(g_1 + g_2) \leqslant 1 .$$

It is seen that a supporting hyperplane $\langle \pi^-, v \rangle \geqslant 1$ of P that induces a facet has a very special algebraic structure as a function of the group elements of G. It will be shown that any facet induced by a supporting hyperplane of the form $\langle \pi^+, v \rangle \leqslant 1$ is the complementary facet of a facet induced by a hyperplane of the form $\langle \pi^-, v \rangle \geqslant 1$ or $\langle \pi^\circ, v \rangle \geqslant 0$.

Lemma 3.6. *Let $\langle \pi^+, \mathrm{v} \rangle \leqslant 1$ be a supporting hyperplane of a collapsed kernel polyhedron* P *that induces a facet* F*. Then* F *is the complement of a facet induced by a supporting hyperplane of the form $\langle \pi^-, \mathrm{v} \rangle \geqslant 1$ or $\langle \pi^\circ, \mathrm{v} \rangle \geqslant 0$.*

Proof. Suppose F is any j-dimensional face of P, induced by a supporting hyperplane π. There are three cases to consider. Suppose π is of the form $\langle \pi^+, v\rangle \leqslant 1$. Let v^c denote the complementary vector and note that complementation maps P onto itself and carries the vertex v of P onto another vertex, v^c of P. The following relations are evident:

$$\langle \pi^+, v\rangle \leqslant 1\,,$$

$$\langle \pi^+, v^c\rangle \leqslant 1\,,$$

$$\langle \pi^+, v + v^c\rangle = \sum_{g\in G^+} \pi^+(g)O(g)\,,$$

$$\langle \pi^+, v^c\rangle = \sum_{g\in G^+} \pi^+(g)O(g) - \langle \pi^+, v\rangle \geqslant \sum_{g\in G^+} \pi^+(g)O(g) - 1\,,$$

$$\langle \pi^+, v\rangle \geqslant \sum_{g\in G^+} \pi^+(g)O(g) - 1\,.$$

In fact, the face induced by the supporting hyperplane

$$\langle \pi^+, v\rangle \geqslant \sum_{g\in G^+} \pi^+(g)O(g) - 1$$

is the complement of the face induced by the supporting hyperplane $\langle \pi^+, v\rangle \leqslant 1$. It is clear that these faces are parallel and disjoint. Note that this proves there are no proper self-complementary faces of P.

For supporting hyperplanes of the form $\langle \pi^-, v\rangle \geqslant 1$ or $\langle \pi^\circ, v\rangle \geqslant 0$, the arguments are analogous. The following is a list of the associated supporting hyperplanes:

$$\langle \pi^-, v\rangle \geqslant 1\,,$$

$$\langle \pi^-, v\rangle \leqslant \sum_{g\in G^+} \pi^-(g)O(g) - 1\,,$$

$$\langle \pi^\circ, v\rangle \geqslant 0\,,$$

$$\langle \pi^\circ, v\rangle \leqslant \sum_{g\in G^+} \pi^\circ(g)O(g)\,.$$

Suppose F is a facet induced by a supporting hyperplane of the form $\langle \pi^+, v\rangle \leqslant 1$. Consider the associated parallel half-space

$$\langle \pi^+, v \rangle \geqslant \sum_{g \in G^+} \pi^+(g) O(g) - 1 .$$

If

$$\lambda = \sum_{g \in G^+} \pi^+(g) O(g) - 1 \geqslant 0 ,$$

define $\pi^* = (1/\lambda)\pi^+$ when $\lambda \neq 0$ and $\pi^* = \pi^+$ when $\lambda = 0$. Then π^* induces a complementary facet of the form stated in the theorem. It is sufficient to show that the case $\lambda < 0$ cannot hold.

Suppose the contrary. There exists a facet F induced by a supporting hyperplane of the form $\langle \pi^+, v \rangle \leqslant 1$, and $\lambda = \Sigma_{g \in G^+} \pi^+(g) O(g) - 1$ is negative. For any component g', there is a vertex v' of F such that $v'(g') < O(g')$. Otherwise, for every vertex v of F, $v(g') = O(g')$ and since the set of vertices of a facet determines the supporting hyperplane, $\pi^+(g) = 0$ for $g \neq g'$ and $\pi^+(g') = 1/O(g')$. Then

$$\lambda = \frac{1}{O(g')} O(g') - 1 = 0 .$$

It is possible to construct a vector v^* in F such that $v^*(g) < O(g)$ for every component g. This construction is based on the following observations. Let v^1 and v^2 be two vectors in F and define

$$v^*(g) = v^1(g) + v^2(g) \text{ Mod } O(g) .$$

Note that any vector v° whose gth components are $O(g)$ or 0 satisfies $\langle \pi^+, v^\circ \rangle \leqslant 1$. The following sequence of relations is evident:

$$\langle \pi^+, v^1 + v^2 \rangle = \langle \pi^+, v^* \rangle + \langle \pi^+, v^\circ \rangle = 2 ,$$

$$\langle \pi^+, v^\circ \rangle \leqslant 1 ,$$

$$\langle \pi^+, v^* \rangle \geqslant 1 ,$$

$$\langle \pi^+, v^* \rangle \leqslant 1 ,$$

$$\langle \pi^+, v^* \rangle = 1 .$$

By an inductive construction on the components, a vector v^* in F of the desired form can be constructed.

Next define a vector v^3 as

$$v^3(g) = v^*(g) \qquad g \neq g_0 \text{ or } -g_0,$$

$$v^3(g_0) = v^*(g_0) + 1,$$

$$v^3(-g_0) = v^*(-g_0) + 1.$$

It is clear that v^3 is in P and consequently $\langle \pi^+, v^3 \rangle \leqslant 1 = \langle \pi^+, v^* \rangle$.

The following inequality is a consequence of the above construction;

$$\pi^+(g) + \pi^+(-g) \leqslant 0 \text{ for all } g \in G^+.$$

The next inequality follows in a similar manner. Define a vector v^4 as

$$v^4(g) = v^*(g) \qquad g \neq g_1 + g_2, -g_1, \text{ or } -g_2,$$

$$v^4(g_1 + g_2) = v^*(g_1 + g_2) + 1,$$

$$v^4(-g_1) = v^*(-g_1) + 1,$$

$$v^4(-g_2) = v^*(-g_2) + 1.$$

Thus,

$$\text{(I)} \qquad \pi^+(g_1 + g_2) + \pi^+(-g_1) + \pi^+(-g_2) \leqslant 0.$$

Consider the original supporting hyperplane

$$\langle \pi^+, v \rangle \geqslant \sum_{g \in G^+} \pi^+(g) O(g) - 1,$$

which induces the complementary facet. Recall that $\lambda = \Sigma_{g \in G^+} \pi^+(g) O(g) - 1$ is negative and consequently this supporting hyperplane can be expressed in the form $\langle (1/\lambda)\pi^+, v \rangle \leqslant 1$. By an argument similar to the above, it is found that

$$\frac{1}{\lambda}\pi^+(g) + \frac{1}{\lambda}\pi^+(-g) \leqslant 0 \text{ for all } g \in G^+.$$

Since λ is negative, we have

$$\text{(II)} \qquad \pi^+(g) + \pi^+(-g) \geqslant 0 \,,$$

$$\pi^+(g) + \pi^+(-g) \leqslant 0 \,,$$

$$\pi^+(g) + \pi^+(-g) = 0 \qquad \text{all } g \in G^+ \,.$$

Using conditions (I) and (II), we obtain

$$\pi^+(g_1 + g_2) = \pi^+(g_1) + \pi^+(g_2) \,.$$

These relations imply that

$$\pi^+(g) + \pi^+(-g) = \pi^+(g) + \pi^+([O(g) - 1]g)$$

$$= \pi^+(g) + [O(g) - 1]\pi^+(g) = O(g)\pi^+(g) = 0 \,.$$

Hence, $\pi^+(g) = 0$ for all g. This is a contradiction. There are no facets induced by a supporting hyperplane of the form $\langle \pi^+, v \rangle \leqslant 1$ and $\Sigma\, \pi^+(g)O(g) < 1$.

The above discussion indicates that the facets of a collapsed kernel polyhedron associated with an Abelian group G are determined by a set of functions π with the properties

$\pi : G \to R$ where R is the real number system,

(i) $\pi(O) = 0$, where O is the zero group element

$\pi(g) \geqslant \dfrac{1}{O(g)}$ where $O(g)$ is the order of g,

(ii) $\pi(g) + \pi(-g) = 1$,

(iii) $\pi(g_1 + g_2) \leqslant \pi(g_1) + \pi(g_2) \leqslant 1 + \pi(g_1 + g_2)$.

Actually, any function π that satisfies the above conditions generates a supporting hyperplane $\langle \pi, v \rangle \geqslant 1$ and induces a proper face of the collapsed kernel polyhedron over the group G. This is apparent from the following relations. Suppose the following equation and inequalities hold for a vector v over a finite group G:

$$\sum_{g \in G^+} v(g)g = 0 \,,$$

$$0 \leqslant v(g) \leqslant O(g) \,,$$

$$0 < v(g_0) \,.$$

Then,

$$-g_0 = [v(g_0) - 1]g_0 + \sum_{g \neq g_0} v(g)g,$$

$$\pi(-g_0) \leqslant \pi(g_0)[v(g_0) - 1] + \sum_{g \neq g_0} \pi(g)v(g),$$

$$\sum_{g \in G^+} \pi(g)v(g) \geqslant \pi(g_0) + \pi(-g_0) = 1,$$

$$\langle \pi, v \rangle \geqslant 1.$$

Consequently, $\langle \pi, v \rangle \geqslant 1$ is a supporting hyperplane of P and induces a face of P.

There are interesting connections between the algebraic behavior of the functions π and the geometric nature of the faces that they induce. These considerations motivate the following definitions and results.

Definition 21. Local morphism

A local morphism ψ is a map from a finite Abelian group G to R/Z. It has the properties

(i) $\psi(O) = 0$, where O and 0 are the zero group elements of G and R/Z, respectively,

(ii) $\psi(-g) = -\psi(g)$,

(iii) For any g_1 there is a nonempty set S_{g_1} where $g_2 \in S_{g_1}$ implies that $\psi(g_1) + \psi(g_2) = \psi(g_1 + g_2)$.

Definition 22. Core of a local morphism

The core of a local morphism ψ is the set of all unordered pairs $[g_1, g_2]$ of elements of G with the property

(i) $\psi(g_1) + \psi(g_2) = \psi(g_1 + g_2)$.

Recall that R/Z denotes the circle group, the factor group of the real numbers under addition modulo the integers. Let λ denote the canonical lifting of R/Z to the interval $[0,1]$. Recall that R/Z can be considered as the collection of cosets $\{x + Z\}$ where $O \leqslant x \leqslant 1$ and addition is performed modulo 1. In this case $\lambda(x + Z) = x$.

Lemma 3.7. *Any nontrivial facet of a collapsed kernel polyhedron of the group* G *induces a local morphism* ψ *of* G. *Any local morphism* ψ *of* G *with the property*

$$\lambda\psi(g_1 + g_2) + \lambda\psi(-g_1) + \lambda\psi(-g_2) \geqslant 1,$$

induces a face of P.

Proof. Any nontrivial facet is associated with a supporting hyperplane of the form $\langle \pi, v \rangle \geqslant 1$. It was shown that π has the properties

(i) $\pi(O) = 0,$

(ii) $\pi(g) + \pi(-g) = 1,$

(iii) $\pi(g_1 + g_2) \leqslant \pi(g_1) + \pi(g_2) \leqslant 1 + \pi(g_1 + g_2).$

Let φ denote the canonical homorphism of R onto the factor group R/Z. Consider the composite map $\psi = \varphi\pi$. Conditions (i) and (ii) assume the forms

(I) $\psi(O) = 0,$

(II) $\psi(g) + \psi(-g) = \varphi(1) = 0.$

Consequently, ψ is a local morphism.

Suppose ψ is a local morphism with the property

$$\lambda\psi(g_1 + g_2) + \lambda\psi(-g_1) + \lambda\psi(-g_2) \geqslant 1.$$

Define $\pi = \lambda\psi$. Then the following relations are evident:

$$\pi(g_1 + g_2) + \pi(-g_1) + \pi(-g_2) \geqslant 1,$$

(i) $\pi(O) = 0;$

$$\pi(O) + \pi(g) + \pi(-g) \geqslant 1,$$

$$\pi(g) + \pi(-g) \geqslant 1,$$

(ii) $\pi(g) + \pi(-g) = 1;$

$$\pi(-g_1 - g_2) + \pi(g_1) + \pi(g_2) \geqslant 1 = \pi(g_1 + g_2) + \pi(-g_1 - g_2),$$

(iiia) $\pi(g_1 + g_2) \leqslant \pi(g_1) + \pi(g_2);$

$$\pi(g_1 + g_2) + \pi(-g_1) + \pi(-g_2) + \pi(g_1) + \pi(g_2) \geqslant 1 + \pi(g_1) + \pi(g_2),$$

$$\pi(g_1 + g_2) + 2 \geqslant 1 + \pi(g_1) + \pi(g_2),$$

(iiib) $\pi(g_1) + \pi(g_2) \leqslant 1 + \pi(g_1 + g_2).$

Consequently, π induces a face of P.

The significance of the concept of local morphism and core elements is more clearly shown in the next result. This result indicates the close connection between the geometric structure of a facet and the algebraic properties of the associated local morphism.

Theorem 7. *Let $\langle \pi, v\rangle \geqslant 1$ be the supporting hyperplane of a facet* F. *Let v be an integer vector in* F *with $v(g_1) > 0$ and $v(g_2) > 0$; then $\pi(g_1) + \pi(g_2) = \pi(g_1 + g_2)$. Furthermore, the vector v^* is also in* F, *where v^* satisfies*

$$v^*(g) = v(g) \qquad g \neq g_1 + g_2, g_1, \text{ or } g_2,$$

$$v^*(g_1 + g_2) = v(g_1 + g_2) + 1,$$

$$v^*(g_1) = v(g_1) - 1,$$

$$v^*(g_2) = v(g_2) - 1.$$

In addition, if $v(g^\circ) > 0$, $g^\circ = g_1 + g_2$, and $\pi(g^\circ) = \pi(g_1) + \pi(g_2)$, then $\bar{v}$ is in F, *where $\bar{v}$ satisfies*

$$\bar{v}(g) = v(g) \qquad g \neq g_1 + g_2, g_1, \text{ or } g_2,$$

$$\bar{v}(g_1 + g_2) = v(g_1 + g_2) - 1,$$

$$\bar{v}(g_1) = v(g_1) + 1,$$

$$\bar{v}(g_2) = v(g_2) + 1.$$

Proof. Consider

$$\langle \pi, v\rangle = \sum_{g \in G^+} \pi(g)v(g) = 1$$

$$= \pi(g_1 + g_2)v(g_1 + g_2) + \pi(g_1)v(g_1) + \pi(g_2)v(g_2) + \sum \pi(g)v(g).$$

If the definition of a collapsed kernel polyhedron over a group G and some elementary algebra are used, the conclusions of the theorem are easily derived.

Corollary 7.1. *Let* F *be a facet induced by the supporting hyperplane $\langle \pi, v\rangle \geqslant 1$. If v^* is a vertex of* F *such that $v^*(g_0) > 1$, then for all vertices v of* F, $v(-g_0)$ *is 0 or 1.*

Proof. Suppose there exists a vertex $\bar{v}$ of F such that $\bar{v}(-g_0) > 1$. Consider the relation

$$\langle \pi, v^*\rangle + \langle \pi, \bar{v}\rangle = \pi(g_0)v^*(g_0) + \sum_{g \neq g_0} \pi(g)v^*(g) + \pi(-g_0)\bar{v}(-g_0)$$

$$+ \sum_{g \neq -g_0} \pi(g)\bar{v}(g) = 2.$$

However,

$$\pi(g_0)v^*(g_0) + \pi(-g_0)\bar{v}(-g_0) \geqslant 2.$$

Thus, we obtain a contradiction and this establishes the corollary.

In conclusion, the above results offer but a glimpse of some of the interesting algebraic, geometric, and computational problems associated with the group theoretic approach to linear integer programming. It is clear that there are a number of interesting questions which are unresolved. For example, can a precise geometric characterization of Abelian polyhedra be determined? What is the exact connection between the algebraic structure of an Abelian group and the geometric structure of its associated Abelian polyhedra? How can the special structure of Abelian polyhedra be exploited to develop efficient algorithms for solving linear integer programs on a computer?

ACKNOWLEDGMENTS

I wish to express my appreciation to my scientific advisor, Len Weiss, for his help and support, to Peggy Kost for her patience in preparing the final manuscript, and to Cathy, Ben, and T. J. for their constant encouragement.

This research was supported by a Resident Research Associateship jointly sponsored by the Naval Research Laboratory and the National Research Council of the National Academy of Sciences.

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Appendix A

COMPUTER LISTING

The group theoretic method for solving linear integer programs has been implemented on a computer. The following is a listing of an implementation on the CDC 6600 computer. The program is written in Fortran IV and runs under the Kronos Time-Sharing System. Following the program listings is an example problem that was solved using the program.

Consider the following example of a linear integer program:

Maximize $Z = 4x_1 + 5x_2 + x_3$ subject to $3x_1 + 2x_2 \leqslant 10$, $x_1 + 4x_2 \leqslant 11$, $3x_1 + 3x_2 + x_3 \leqslant 13$, $x_1, x_2, x_3 \geqslant 0$, and x_1, x_2, and x_3 are integers.

This program was run on the computer, and the following correct results were obtained. It should be cautioned that the computer program is experimental. There is still much work to be done on developing numerical methods for the group theoretic approach to linear integer programming.

$$3x_1 + 2x_2 \leqslant 10$$

$$x_1 + 4x_2 \leqslant 11$$

$$3x_1 + 3x_2 + x_3 \leqslant 13$$

$x_1, x_2, x_3 \geqslant 0$ and $x_1, x_2,$ and x_3 are integers,

```
 74/06/25. 11.06.21.
PROGRAM    BEN

      PROGRAM GROUP(INPUT,OUTPUT)
C     INTEGER PROGRAMMING
      COMMON /ALPHA/ A(60,30)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
     DIMENSION ISOL(100)
      DOUBLE PRECISION A,AMIN,ETA
      PRINT 1000
      READ,M,N
      ETA = -1.0D-3
      ZETA = 2**9
      MN = M + N + 2
      MN1 = MN - 1
      NP = N + 1
      PRINT 1010
      DO 10 J=1,NP
      A(J,J) = -1.0D0
      A(MN,J) = -1.0D0
      READ,C
 1100 FORMAT(F10.5)
      IF(J.EQ.1) GO TO 10
      IA(M+2,J-1) = IFIX(C)
      IA(M+1,J-1) = 1
   10 A(1,J) = -C
      A(1,1) = -A(1,1)
      IA(M+2,MN) = IFIX(A(1,1))
      PRINT 1020
      DO 20 I=1,M
      READ,B
      IA(I,N+I) = 1
      IA(I,MN) = IFIX(B)
      II = I + NP
   20 A(II,1) = B
      A(MN,1) = ZETA
      PRINT 1030
      IA(M+1,MN1) = 1
      IA(M+1,MN) = IFIX(ZETA)
      READ,ICT
      DO 30 II=1,ICT
      READ,I,J,AX
      IA(I,J) = IFIX(AX)
      I = I + NP
      J = J + 1
   30 A(I,J) = AX
  900 FORMAT(I10)
C     COLUMN ORDER AND BASIS VECTORS
      DO 32 J=1,MN
   32 IA(M+3,J) = J
      M1 = M + 1
      DO 35 J=1,M1
   35 IB(N+J) = 1
C     INITIAL PIVOT ON LEXICO. SMALLEST COLUMN
      J = LEXCO(MN)
```

```
      IF(J.EQ.0) GO TO 100
      DO 40 I=1,MN1
      IF(A(I,J).NE.0.0D0) GO TO 50
   40 CONTINUE
   50 IF(A(I,J).LT.0.0D0) GO TO 60
      GO TO 80
   60 DO 70 K=2,NP
   70 A(MN,K) = 1.0D0
      CALL PIVOT(MN,J)
      IB(MN1) = 0
      IB(J-1) = 1
C     DUAL SIMPLEX
   80 AMIN = 1.0D300
      DO 90 I=2,MN
      IF(A(I,1).GE.AMIN) GO TO 90
      AMIN = A(I,1)
      IMIN = I
   90 CONTINUE
      I = IMIN
      IF(AMIN.GE.ETA) GO TO 100
      J = LEXCO(I)
      IF(J.EQ.0) GO TO 100
      IB(I-1) = 0
      K = NBAS(J)
      IB(K) = 1
      CALL PIVOT(I,J)
      GO TO 80
C     OPTIMAL SOLUTION PRINT-OUT
  100 PRINT 1050,A(1,1)
      DO 110 I=2,NP
      II = I - 1
      IF(IB(II).EQ.1) PRINT 1080,II,A(I,1)
      IF(IB(II).EQ.1) GO TO 110
      PRINT 1060,II,A(I,1)
  110 CONTINUE
      ILO = NP + 1
      DO 120 I=ILO,MN
      II = I - NP
      IF(IB(I-1).EQ.1) PRINT 1090,II,A(I,1)
      IF(IB(I-1).EQ.1) GO TO 120
      PRINT 1070,II,A(I,1)
  120 CONTINUE
C     PERMUTE BASIS INTO FIRST M+1 COLUMNS
      JC = 0
      M3 = M + 3
      DO 140 J=1,MN1
      IF(IB(J).EQ.0) GO TO 140
      JC = JC + 1
      DO 130 I=1,M3
      ITEMP = IA(I,JC)
      IA(I,JC) = IA(I,J)
  130 IA(I,J) = ITEMP
  140 CONTINUE
      IE = 3
```

```
      CALL SFORM(JC)
      CALL EROR(IE)
      CALL SETUP(JC)
      IE = 4
      CALL MIN(JC)
      CALL EROR(IE)
      CALL XTRAK(ISOL)
      CALL SOLVE(ISOL)
      STOP
 1000 FORMAT(3X,16HTYPE MATRIX SIZE)
 1010 FORMAT(3X,16HTYPE COST VECTOR)
 1020 FORMAT(3X,23HTYPE REQUIREMENT VECTOR)
 1030 FORMAT(3X,29HTYPE NO. OF NON-ZERO ELEMENTS)
 1040 FORMAT(5X,2I5,F10.5)
 1050 FORMAT(3X,10HMAXIMUM = ,D20.10)
 1060 FORMAT(6X,I3,2X,D20.10)
 1070 FORMAT(6X,I3,1HS,1X,D20.10)
 1080 FORMAT(6X,I3,2X,D20.10,7H*BASIC*)
 1090 FORMAT(6X,I3,1HS,1X,D20.10,7H*BASIC*)
      END
      FUNCTION LEXCO(I)
      COMMON /ALPHA/ A(60,30)
      COMMON /BETA/ IDETR,M,N
      DIMENSION JX(60,2)
      DOUBLE PRECISION A,X,XMAX
C     INITIAL COLUMN SEARCH
      MN1 = M + N + 1
      NP = N + 1
      XMAX = -1.0D250
      I0 = 1
      J0 = 0
      IC = 1
      IP = 2
      DO 10 J=2,NP
      IF(A(I,J).GE.0.0D0) GO TO 10
      X = A(1,J)/A(I,J)
      IF(X.LT.XMAX) GO TO 10
      IF(X.GT.XMAX) J0 = 0
      XMAX = X
      J0 = J0 + 1
      JX(J0,1) = J
   10 CONTINUE
      IM1 = I - 1
      IF(J0.EQ.0) PRINT 1000,IM1
 1000 FORMAT(3X,37HINFEASIBLE CONDITION DETECTED AT ROW ,I5)
      IF(J0.EQ.0) JX(1,IC) = 0
      IF(J0.EQ.0) GO TO 40
C     FIND LEXICO. GREATEST COLUMN
   20 IF(J0.EQ.1) GO TO 40
      I0 = I0 + 1
      IF(I0.GT.MN1) PRINT 1010
 1010 FORMAT(6X,25HERROR 2 IDENTICAL COLUMNS)
      IF(I0.GT.MN1) STOP
      XMAX = -1.0D250
```

```
      JUP = J0
      J0 = IP
      IP = IC
      IC = J0
      J0 = 0
      DO 30 J=1,JUP
      K = JX(J,IP)
      X = A(I0,K)/A(I,K)
      IF(X.LT.XMAX) GO TO 30
      IF(X.GT.XMAX) J0 = 0
      XMAX = X
      J0 = J0 + 1
      JX(J0,IC) = K
   30 CONTINUE
      GO TO 20
   40 LEXCO = JX(1,IC)
      RETURN
      END
      SUBROUTINE PIVOT(I,J)
      COMMON /ALPHA/ A(60,30)
      COMMON /BETA/ IDETR,M,N
      DOUBLE PRECISION A,P
      MN = M + N + 2
      NP = N + 1
      P = (1.0D0/A(I,J))*(-1.0D0)
      DO 10 II=1,MN
   10 A(II,J) = P*A(II,J)
      A(I,J) = -1.0D0
      DO 30 JJ=1,NP
      IF(JJ.EQ.J) GO TO 30
      P = A(I,JJ)
      DO 20 II=1,MN
   20 A(II,JJ) = A(II,JJ) + P*A(II,J)
      A(I,JJ) = 0.0D0
   30 CONTINUE
      RETURN
      END
      SUBROUTINE SFORM(JC)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
C     DIAGONALIZE BASIS
      M1 = M + 1
      MN = M + N + 2
   10 JC = 1
   20 CALL SMAL(JC)
   30 DO 40 J=JC,M1
      IF(MOD(IA(JC,J),IA(JC,JC)).NE.0) GO TO 50
   40 CONTINUE
      GO TO 70
   50 NQ = IA(JC,J)/IA(JC,JC)
      DO 60 I=JC,M1
   60 IA(I,J) = IA(I,J) - NQ*IA(I,JC)
      GO TO 20
   70 DO 80 I=JC,M1
```

```
      IF(MOD(IA(I,JC),IA(JC,JC)).NE.0) GO TO 90
   80 CONTINUE
      GO TO 110
   90 NQ = IA(I,JC)/IA(JC,JC)
      MN = M + N + 2
      DO 100 J=JC,MN
  100 IA(I,J) = IA(I,J) - NQ*IA(JC,J)
      GO TO 20
  110 JC1 = JC + 1
      DO 120 J=JC1,M1
      NQ = IA(JC,J)/IA(JC,JC)
      DO 120 I=JC,M1
  120 IA(I,J) = IA(I,J) - NQ*IA(I,JC)
      DO 130 I=JC1,M1
      NQ = IA(I,JC)/IA(JC,JC)
      DO 130 J=JC,MN
  130 IA(I,J) = IA(I,J) - NQ*IA(JC,J)
      IF(JC.EQ.M) GO TO 140
      JC = JC + 1
      GO TO 20
C     FORM PROPER DIVISORS
  140 JC = 1
  150 CALL SMAL(JC)
      DO 160 I=JC,M1
      IF(MOD(IA(I,I),IA(JC,JC)).NE.0) GO TO 170
  160 CONTINUE
      GO TO 200
  170 NQ = IA(I,I)/IA(JC,JC)
      NR = IA(I,I) - NQ*IA(JC,JC)
      DO 180 J=JC,MN
  180 IA(I,J) = IA(I,J) + NQ*IA(JC,J)
      IA(JC,JC) = -IA(JC,JC)
      IA(I,JC) = NR
      IA(JC,I) = -IA(JC,JC)
      IA(I,I) = NQ*IA(JC,I)
      DO 190 J=JC,MN
      ITEMP = IA(JC,J)
      IA(JC,J) = IA(I,J)
  190 IA(I,J) = ITEMP
      GO TO 10
  200 IF(JC.EQ.M) RETURN
      JC = JC + 1
      GO TO 150
      RETURN
      END
      SUBROUTINE SMAL(JC)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      IMIN = JC
      JMIN = JC
      M1 = M + 1
      MIN = 2**47
      MN = M + N + 2
      DO 10 I=JC,M1
```

```
      DO 10 J=JC,M1
      IF(IABS(IA(I,J)).EQ.0) GO TO 10
      IF(IABS(IA(I,J)).GE.MIN) GO TO 10
      IMIN = I
      JMIN = J
      MIN = IABS(IA(I,J))
   10 CONTINUE
C     PERMUTE ROW JC AND IMIN
      DO 20 J=JC,MN
      ITEMP = IA(JC,J)
      IA(JC,J) = IA(IMIN,J)
   20 IA(IMIN,J) = ITEMP
C     PERMUTE COLUMN JC AND JMIN
      M3 = M + 3
      DO 30 I=JC,M3
      ITEMP =  IA(I,JC)
      IA(I,JC) = IA(I,JMIN)
   30 IA(I,JMIN) = ITEMP
      RETURN
      END
      SUBROUTINE SETUP(J1)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      I'MENSION IVEC(100)
C     NUN-BASIC MODULO BASIC
      J1 = M + 2
      J2 = M + N + 2
      M1 = M + 1
      DO 10 I=1,M1
      DO 10 J=J1,J2
      IA(I,J) = MOD(IA(I,J),IABS(IA(I,I)))
   10 IF(IA(I,J).LT.0) IA(I,J) = IA(I,J) + IABS(IA(I,I))
C     INITIALIZE IGRP FOR GROUP MINIMIZATION
      IDETR = 1
      DO 20 I=1,M1
   20 IDETR = IDETR*IABS(IA(I,I))
      DO 30 I=1,IDETR
      IGRP(I,1) = 0
      IGRP(I,2) = 0
      IGRP(I,3) = I
   30 CST(I) = 1.0E100
C     ENTER NON-BASIC INTO GROUP TABLE
      J2 = M + N + 1
      DO 50 J=J1,J2
      DO 40 I=1,M1
   40 IVEC(I) = IA(I,J)
      JA = IADR(IVEC)
      RC = RCST(IA(M+3,J))
      IF(RC.LT.CST(JA)) IGRP(JA,1) = IA(M+3,J)
   50 CST(JA) = AMIN1(CST(JA),RC)
      CST(1) = 0.0
      RETURN
      END
      FUNCTION RCST(I)
```

```
      COMMON /ALPHA/ A(60,30)
      COMMON /BETA/ IDETR,M,N
      DOUBLE PRECISION A
      N1 = N + 1
      DO 10 J=2,N1
      K = J
      IF(A(I+1,J).LT.(-.5D0)) GO TO 20
   10 CONTINUE
   20 RCST = A(1,K)
      RETURN
      END
      SUBROUTINE MIN(ILB)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      DIMENSION IVEC1(60),IVEC2(60)
C     GROUP MINIMIZATION
      ILB = 1
      M1 = M + 1
   10 COST = 1.0E300
      IMIN = 1
      DO 20 I=2,IDETR
      IF(IGRP(I,2).EQ.1) GO TO 20
      IF(CST(I).GE.COST) GO TO 20
      COST = CST(I)
      IMIN = I
   20 CONTINUE
      IGRP(IMIN,2) = 1
      CALL INVAD(IVEC1,IMIN)
      DO 40 I=2,IDETR
      IF(IGRP(I,2).EQ.0) GO TO 40
      CALL INVAD(IVEC2,I)
      DO 30 J=1,M1
      IVEC2(J) = IVEC2(J) + IVEC1(J)
   30 IVEC2(J) = MOD(IVEC2(J),IABS(IA(J,J)))
      IAD = IADR(IVEC2)
      IF((CST(I)+CST(IMIN)).GE.CST(IAD)) GO TO 40
     CST(IAD) = CST(I) + CST(IMIN)
      IGRP(IAD,3) = IGRP(I,3)
   40 CONTINUE
      ILB = ILB + 1
      IF(ILB.LT.IDETR) GO TO 10
      RETURN
      END
      SUBROUTINE XTRAK(ISOL)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      DIMENSION ISOL(100),IVEC1(100),IVEC2(100)
C     EXTRACT NON-BASIC VARIABLES FROM GROUP TABLE
     IE = 1
      M1 = M + 1
      MN = M + N + 2
      DO 10 I=1,M1
   10 IVEC1(I) = IA(I,MN)
      IG1 = IADR(IVEC1)
```

```
   20 IG2 = IGRP(IG1,3)
      IF(IGRP(IG2,1).EQ.0) CALL EROR(IE)
      K = IGRP(IG2,1)
      ISOL(K) = ISOL(K) + 1
      CALL INVAD(IVEC2,IG2)
      DO 30 J=1,M1
      IVEC1(J) = IVEC1(J) + IABS(IA(J,J)) - IVEC2(J)
   30 IVEC1(J) = MOD(IVEC1(J),IABS(IA(J,J)))
      IG1 = IADR(IVEC1)
      IF(IG1.NE.1) GO TO 20
      RETURN
      END
      FUNCTION IADR(IVEC)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      DIMENSION IVEC(60)
      K = IDETR
      KA = 0
      M1 = M + 1
      DO 10 I=1,M1
      K = K/IABS(IA(I,I))
   10 KA = KA + IVEC(I)*K
      IADR = KA + 1
      RETURN
      END
      SUBROUTINE INVAD(IVEC,I)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      DIMENSION IVEC(60)
      K = I - 1
      M1 = M + 1
      DO 10 J=1,M1
      L = M + 2 - J
      IVEC(L) = MOD(K,IABS(IA(L,L)))
   10 K = (K - IVEC(L))/IABS(IA(L,L))
      RETURN
      END
      SUBROUTINE SOLVE(ISOL)
      COMMON /ALPHA/ A(60,30)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      DOUBLE PRECISION A,S
      DIMENSION ISOL(100)
      MN = M + N + 2
      N1 = N + 1
      DO 30 I=1,MN
      IF(I.EQ.1) GO TO 10
      IF(IB(I-1).EQ.0) GO TO 30
   10 S = A(I,1)
      DO 20 J=2,N1
      K = NBAS(J)
   20 S = S - A(I,J)*DBLE(FLOAT(ISOL(K)))
      IF(S.GE.0.0D0) S = S + .5D0
      IF(S.LT.0.0D0) S = S - .5D0
```

```
      IF(I.EQ.1) IMAX = IDINT(S)
      IF(I.EQ.1) GO TO 30
      ISOL(I-1) = IDINT(S)
   30 CONTINUE
      PRINT 1000,IMAX
      DO 40 I=1,N
      IF(IB(I).EQ.0) PRINT 1010,I,ISOL(I)
      IF(IB(I).EQ.1) PRINT 1020,I,ISOL(I)
   40 CONTINUE
      MN1 = MN - 1
      DO 50 I=N1,MN1
      IF(IB(I).EQ.0) PRINT 1030,I,ISOL(I)
      IF(IB(I).EQ.1) PRINT 1040,I,ISOL(I)
   50 CONTINUE
 1000 FORMAT(3X,18HINTEGER MAXIMUM = ,I10)
 1010 FORMAT(6X,I3,2X,I10)
 1020 FORMAT(6X,I3,2X,I10,7H*BASIC*)
 1030 FORMAT(6X,I3,1HS,1X,I10)
 1040 FORMAT(6X,I3,1HS,1X,I10,7H*BASIC*)
      RETURN
      END
      FUNCTION NBAS(K)
      COMMON /ALPHA/ A(60,30)
      COMMON /BETA/ IDETR,M,N
      DOUBLE PRECISION A
      IE = 2
      MN = M + N + 2
      N1 = N + 1
      DO 20 I=2,MN
      IF(A(I,K).NE.(-1.0D0)) GO TO 20
      DO 10 J=1,N1
      IF(J.EQ.K) GO TO 10
      IF(A(I,J).NE.0.0D0) GO TO 20
   10 CONTINUE
      NBAS = I - 1
      GO TO 30
   20 CONTINUE
      CALL EROR(IE)
   30 RETURN
      END
      SUBROUTINE EROR(IE)
      COMMON /BETA/ IDETR,M,N
      COMMON /GAMMA/ CST(500),IA(60,100),IB(100),IGRP(500,3)
      MN = M + N + 2
      M1 = M + 1
      GO TO (10,20,30,40),IE
   10 PRINT 1000
      GO TO 50
   20 PRINT 1010
      GO TO 50
   30 PRINT 1020
      PRINT 1025,(IA(I,I),I=1,M1)
      GO TO 50
   40 PRINT 1030,(I,CST(I),IGRP(I,1),IGRP(I,2),IGRP(I,3),I=1,IDETR)
```

```
 1000 FORMAT(3X,25HERROR IN SUBROUTINE XTRAK/
+      3X,26HGROUP MINIMIZATION PROBLEM)
 1010 FORMAT(3X,24HERROR IN SUBROUTINE NBAS/3X,
+      33HTROUBLE LOCATING NON-BASIC VECTOR)
 1020 FORMAT(10X,14HINTEGER MATRIX///)
 1025 FORMAT(15I5)
 1030 FORMAT(10X,24HGROUP MINIMIZATION TABLE///(3X,I3,3X,F10.5,3X,I5,
+     2X,I5,2X,I5))
   50 RETURN
      END
READY.
```

```
RUN

 74/06/25. 11.19.23.
PROGRAM   BEN

   TYPE MATRIX SIZE
? 3,3
   TYPE COST VECTOR
? 0.0
? 4.0
? 5.0
? 1.0
   TYPE REQUIREMENT VECTOR
? 10.0
? 11.0
? 13.0
   TYPE NO. OF NON-ZERO ELEMENTS
? 7
? 1,1,3.0
? 1,2,2.0
? 2,1,1.0
? 2,2,4.0
? 3,1,3.0
? 3,2,3.0
? 3,3,1.0
   MAXIMUM =        .1940000000D+02
        1        .1800000000D+01*BASIC*
        2        .2300000000D+01*BASIC*
        3        .7000000000D+00*BASIC*
        1S          0.
        2S          0.
        3S          0.
        4S       .5072000000D+03*BASIC*
          INTEGER MATRIX


    1     1     1  -10
         GROUP MINIMIZATION TABLE


     1      0.            6      0      1
     2       .20000       4      1      2
     3       .40000       0      1      2
     4       .60000       0      1      2
     5       .80000       0      1      2
     6      1.00000       0      1      2
     7      1.20000       0      1      2
     8       .40000       5      1      8
     9       .60000       0      1      2
    10       .80000       0      1      2
   INTEGER MAXIMUM =          19
        1          2*BASIC*
        2          2*BASIC*
        3          1*BASIC*
        4S         0
        5S         1
        6S         0
        7S       507*BASIC*
    STOP
```